Joe Dinius
Midterm

Math 575B
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1. For the initial value problem

$$\begin{aligned} Y' &= \lambda Y & (1) \\ Y(0) &= Y_0 & (2) \end{aligned}$$

determine the value of α so that the method

$$y_n + \alpha y_{n-1} - (1+\alpha) y_{n-2} = \frac{1}{2}(-\alpha f_n + (4+3\alpha) f_{n-1}) \quad (3)$$

is consistent, convergent, A-stable, and of second order.

Solution. Begin with consistency conditions:

$$\begin{aligned} \sum_{j=0}^{1} a_j &= (1+\alpha) - \alpha = 1 & (4) \\ -\sum_{j=0}^{1} j a_j + \sum_{j=-1}^{0} b_j &= -(1+\alpha) + \frac{1}{2}(-\alpha + (4+3\alpha)) & (5) \\ &= -1 - \alpha - \frac{\alpha}{2} + 2 + \frac{3\alpha}{2} & (6) \\ &= 1 & (7) \\ \sum_{j=0}^{1} (-j)^2 a_j + 2\sum_{j=-1}^{0} (-j)^1 b_j &= (1+\alpha) + 2(-\frac{\alpha}{2}) & (8) \\ &= 1 & (9) \end{aligned}$$

So all consistency conditions are met up to order 2, i.e. method is consistent. Now we need to pick α such that the truncation error is order 2.

The truncation error at the n^{th} step is:

$$\tau_n = \frac{y_n + \alpha y_{n-1} - (1+\alpha) y_{n-2}}{h} - \frac{1}{2}(-\alpha f_n + (4+3\alpha) f_{n-1}) \quad (10)$$

Taylor expand all y's and f's about the n^{th} step and combine. The y_n terms cancel automatically $(1+\alpha-(1+\alpha)=0)$. Now, pick α so that the zeroth order term (y_n') cancels. This yields:

$$\begin{aligned} -\alpha + 2(1+\alpha) + \frac{\alpha}{2} + \frac{4+3\alpha}{2} &= 4 + 3\alpha = 0 & (11) \\ \implies \alpha &= -\frac{4}{3} & (12) \end{aligned}$$

Check that the hy_n'' term cancels:

$$(-\frac{2}{3} + \frac{2}{3} + 0) = 0 \tag{13}$$

So, the first order term cancels, therefore the method is second-order ($\tau_n = -\frac{2}{9}h^2 y_n'''$)
Now the method (3) becomes:

$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = \frac{2}{3}hf_n \tag{14}$$

which is second order.

Now, check the root condition. If the root condition is satisfied, then method is convergent (Dahlquist Equivalence Theorem).

The characteristic equation $(1 - \frac{2}{3}z)r^2 - \frac{4}{3}r + \frac{1}{3})$ has roots $r_{1,2} = \frac{4 \pm 2\sqrt{1+2z}}{6-4z}$. We want to find where:

$$-1 < r_{1,2} < 1 \tag{15}$$

Some messy algebra yields the relations that need to be met:

$$(z-4)(z-\frac{3}{2}) > 0 \tag{16}$$

$$2z(2z-3) > 0 \tag{17}$$

But, these relations are satisfied for all $z \in \mathbb{C}^-$. Therefore the method is A-stable and convergent. Incidentally, this method is known as BDF2.

2. For the IVP

$$Y' = f(x, Y) \tag{18}$$

$$Y(0) = Y_0 \tag{19}$$

There are four basic frameworks for the development of a numerical scheme that approximates the solution(s) of the IVP
a.) What are they?
b.) Pick methods representative of each of these and describe in what sense they belong in each particular framework.

Solution. a.) The four methods are geometric interpretation, Taylor series method, numerical differentiation and numerical integration.

b.) An example of a geometric interpretation method is explicit Euler. Interpret the slope of the tangent line m_n at a point x_n as the derivative of the function we want to find and extrapolate the next point via the relation $y_{n+1} \approx y_n + m_n \Delta x$.

An example of a Taylor series method is any Runge-Kutta numerical scheme. Use the Taylor series expansion for various $Y(x + \alpha\Delta x)$ for $\alpha < 1$ and combine. Derive coefficients that eliminate all lower order terms leaving you with the desired order truncation error.

An example of a numerical differentiation method is the midpoint method. For sufficiently small h, interpret $\frac{Y(x_{n+1})-Y(x_n)}{h} \approx Y'(x_n + \frac{\Delta x}{2}) = f(x_n + \frac{\Delta x}{2}, Y(x_n + \frac{\Delta x}{2}))$. Then, you can determine $Y(x_{n+1}) \approx y_n + \Delta x f(x_n + \frac{\Delta x}{2}, y(x_n + \frac{\Delta x}{2}))$, where y_n denotes the numerical approximation to $Y(x_n)$.

An example of a numerical integration method is an Adams-Bashforth of any order. The given $Y'(x)$ is integrated over each interval and interpolated at however many grid points are desired. That is, $y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx \approx y_n + h[Af_n + Bf_{n-1} + ...]$. The coefficients $A, B, ...$ are determined using the method of undetermined coefficients or Gauss-Jordan elimination.

3. We wish to solve the IVP

$$\begin{aligned} Y' &= f(x, Y) && (20)\\ Y(0) &= Y_0 && (21) \end{aligned}$$

using

$$y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i f(x_{n-i}, y_{n-i}) \tag{22}$$

where $y_n \approx Y(x_n)$, and $n = 1, 2, ...$ For simplicity, assume that the first p values of y_n have been suitably initialized. Suppose we want (21) exact for polynomials of at most m. Use the Method of Undetermined Coefficients to derive the consistency conditions

$$\begin{aligned} \sum_{j=0}^{1} a_j &= 1 \quad i = 0 && (23)\\ -\sum_{j=0}^{1} j a_j + \sum_{j=-1}^{0} b_j &= 1 \quad i = 1 && (24)\\ \sum_{j=0}^{1} (-j)^i a_j + i \sum_{j=-1}^{0} (-j)^{i-1} b_j &= 1 \quad i = 2, 3, ..., m && (25) \end{aligned}$$

Solution. Assume, WLOG, that $h = 1$ and $x_0 = 0$. Since $h = 1$, $x_{n-j} = n - j$ for all $j = 1, 2, ..., m$. Let $w(x)$ be the interpolating polynomial that is exact at all grid points x_j. That is,

$$w(x) = \sum_{i=0}^{m} c_i x^i \tag{26}$$

Rewrite (21) using the fact that $w(x)$ is exact at all grid points up to x_{n-p}

$$y_{n+1} = \sum_{i=0}^{p} a_i w_{n-i} + h \sum_{i=-1}^{p} b_i w'_{n-i} \tag{27}$$

But, since (26) is a linear function in w_n and w_n' and it is exact, this implies (26) must also be exact for all basis functions of the form

$$w_j(x) = x^j \quad j = 0, 1, 2, ..., m \tag{28}$$

Using the first two basis functions and looking at the $n = 0$ point, we have

$$w_0(x_1) = 1 \quad = \quad \sum_{i=0}^{p} a_i w_{0_{0-i}} + \sum_{i=-1}^{p} b_i w'_{0_{0-i}} = \sum_{i=0}^{p} a_i w_{0_{-i}} = \sum_{i=0}^{p} a_i \tag{29}$$

$$w_1(x_1) = x_1 = 1 \quad = \quad \sum_{i=0}^{p} a_i w_{1_{-i}} + \sum_{i=-1}^{p} b_i w'_{1_{-i}} \tag{30}$$

Using the fact that $w_0(x)' = 0$. Now, use the facts that $w_{1_{-i}} = -i$ and $w'_{1_{-i}} = 1$ to get

$$1 \quad = \quad \sum_{i=0}^{p} a_i(-i) + \sum_{i=-1}^{p} b_i \tag{31}$$

$$= \quad -\sum_{i=0}^{p} i a_i + \sum_{i=-1}^{p} b_i \tag{32}$$

For the third consistency condition, let $2 \leq k \leq m$. $w_{k_{-i}} = (-i)^k$ and $w'_{k_{-i}} = k(-i)^{k-1}$

$$w_k(x_1) = x_1^k = 1^k = 1 \quad = \quad \sum_{i=0}^{p} a_i(-i)^k + \sum_{i=-1}^{p} b_i k(-i)^{k-1} \tag{33}$$

$$= \quad \sum_{i=0}^{p} (-i)^k a_i + k \sum_{i=-1}^{p} (-i)^{k-1} b_i \tag{34}$$

which shows that (33) is valid for all $2 \leq k \leq m$. For $k > m$, the polynomial vanishes, i.e. its coefficient is 0, so (33) no longer holds. Therefore the consistency conditions have been derived.

4. Design a convergent BDF1. Show that the problem is consistent, derive its stability range and derive its truncation error.

   Solution. Start with the formula for BDFn and reduce to BDF1

$$y_{n+1} \quad = \quad \sum_{j=0}^{p-1} \alpha_j y_{n-j} + h\beta f(x_{n+1}, y_{n+1}) \tag{35}$$

$$y_{n+1} \quad = \quad \alpha_0 y_n + h\beta f(x_{n+1}, y_{n+1}) \tag{36}$$

   Apply consistency conditions up to order $p = 1$

$$\sum_{j=0}^{0} a_j = \alpha_0 = 1 \tag{37}$$

$$\sum_{j=0}^{0} j a_j + \sum_{j=-1}^{0} = \beta = 1 \tag{38}$$

The method then becomes

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \tag{39}$$

which is implicit Euler.

Apply $f(x, y) = \lambda y$

$$\begin{aligned}
(1 + h\lambda)y_{n+1} &= y_n && (40)\\
y_{n+1} = \frac{y_n}{1 + h\lambda} & && (41)\\
y_{n+1} = \frac{y_0}{(1 + h\lambda)^n} & && (42)
\end{aligned}$$

Letting $z = h\lambda$, and finding the roots of the characteristic equation $r^2 - \frac{1}{1+z}r = 0$ yields

$$\begin{aligned}
r_{1,2} &= \frac{\frac{1}{1+z} \pm \sqrt{\frac{1}{(1+z)^2}}}{2} && (43)\\
r_1 &= \frac{1}{1+z} && (44)\\
r_2 &= 0 && (45)
\end{aligned}$$

$|r_2| = 0 < 1$ obviously satisfies the root condition. For r_1, substitute $z = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$.

$$|r_1| = \sqrt{\frac{1}{(1 + \alpha)^2 + \beta^2}} \tag{46}$$

which is < 1 for all α, β except for $\alpha = -1, \beta = 0$ and $\alpha = 0, \beta = 0$. This implies that the stability range is all of $\mathbb{C}$ except for $z = -1$ and $z = 0$.

The truncation error is

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - f_{n+1} \tag{47}$$

$$\tag{48}$$

Taylor expand y_{n+1} and f_{n+1} about the n^{th} step. The truncation error is

$$\begin{aligned}
\tau_{n+1} &= \frac{hy'_n + \frac{h^2}{2}y''_n + ...}{h} - (y'_n + hy''_n + ...) && (49)\\
&= \frac{h}{2}y''_n - hy''_n + \mathcal{O}(h^2) && (50)\\
&= -\frac{h}{2}y''_n + \mathcal{O}(h^2) && (51)
\end{aligned}$$

5. Take the IVP (1)(2). Assume that the real part of $\lambda < 0$. Suppose we use the trapezoidal method (TRAP) and implicit Euler (IE) to approximate its solutions. Take $h = h_0$ finite. By examining the truncation error (and assuming that the leading order term dominates) infer what qualitative or quantitative differences will be apparent in the solution. The point here is that instead of solving (1)(2), we are solving actually a slightly different equation, depending on whether we use TRAP or IE. What are these different equations and how do the solutions differ from those of (1)(2)? You can use analytical or numerical means here to make your point.

Solution. (TRAP)

$$y_{n+1} = y_n + \frac{h_0\lambda}{2}(y_n + y_{n+1}) \tag{52}$$

$$y_{n+1} = (\frac{(1+\frac{h_0\lambda}{2})}{(1-\frac{h_0\lambda}{2})})^{n+1} y_0 \tag{53}$$

The roots of the characteristic equation are $r_2 = 0$ and $r_1 = \frac{1+\frac{h_0\lambda}{2}}{1-\frac{h_0\lambda}{2}} = -1 + \frac{2}{1-\frac{h_0\lambda}{2}}$, which is guaranteed stable for $\mathrm{Re}[\lambda] < 0$. The truncation error is

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - \frac{1}{2}(f_{n+1} - f_n) \tag{54}$$

$$= (\frac{h_0^2}{6} - \frac{h_0^2}{4})y_n''' + \mathcal{O}(h^3) = -\frac{h_0^2}{12}y_n''' + \mathcal{O}(h_0^3) \tag{55}$$

where the usual method of Taylor series expansion about the n^{th} step is used. But, using $y''' = f'' = \lambda^3 y$, the trapezoid method yields

$$y_{n+1} = y_n + \frac{h_0}{2}(f_{n+1} + f_n) - \frac{h_0^2}{12}\lambda^3 y_n + \mathcal{O}(h_0^3) \tag{56}$$

(IE)
We can use the truncation error from problem 4.

$$\tau_{n+1} = -\frac{h_0}{2}y_n'' + \mathcal{O}(h_0^2) \tag{57}$$

Using $y'' = f' = \lambda^2 y$ the method yields:

$$y_{n+1} = y_n + h_0\lambda y_{n+1} - \frac{h_0}{2}\lambda^2 y_n + \mathcal{O}(h_0^2) \tag{58}$$

Assuming that the leading order terms in the truncation error dominate, it is clear that the numerical schemes solve two different differential equations. The solutions will not differ significantly, except where $h_0\lambda$ is large. However, for $\mathrm{Re}[\lambda] < 0$, the solution y_n will tend towards 0. So, although the methods solve different equations exactly, the solutions will converge to the same solution of the same IVP, just with different truncation error.